SEMISUMMANDS AND SEMIIDEALS IN BANACH SPACES

BY

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ABSTRACT

Let $|\cdot|$ be a fixed absolute norm on $\mathbf{R}^2$. We introduce semi-$|\cdot|$-summands (resp. $|\cdot|$-summands) as a natural extension of semi-L-summands (resp. L-summands). We prove that the following statements are equivalent. (i) Every semi-$|\cdot|$-summand is a $|\cdot|$-summand, (ii) $(1,0)$ is not a vertex of the closed unit ball of $\mathbf{R}^2$ with the norm $|\cdot|$. In particular semi-L^p-summands are L^p-summands whenever $1<p\leqq\infty$. The concept of semi-$|\cdot|$-ideal (resp. $|\cdot|$-ideal) is introduced in order to extend the one of semi-M-ideal (resp. M-ideal). The following statements are shown to be equivalent. (i) Every semi-$|\cdot|$-ideal is a $|\cdot|$-ideal, (ii) every $|\cdot|$-ideal is a $|\cdot|$-summand, (iii) $(0,1)$ is an extreme point of the closed unit ball of $\mathbf{R}^2$ with the norm $|\cdot|$. From semi-$|\cdot|$-ideals we define semi-$|\cdot|$-idealoids in the same way as semi-$|\cdot|$-ideals arise from semi-$|\cdot|$-summands. Proper semi-$|\cdot|$-idealoids are those which are neither semi-$|\cdot|$-summands nor semi-$|\cdot|$-ideals. We prove that there is a proper semi-$|\cdot|$-idealoid if and only if $(1,0)$ is a vertex and $(0,1)$ is not an extreme point of the closed unit ball of $\mathbf{R}^2$ with the norm $|\cdot|$. So there are no proper semi-L^p-idealoids. The paper concludes by showing that w^*-closed semi-$|\cdot|$-idealoids in a dual Banach space are semi-$|\cdot|$-summands, so no new concept appears by predualization of semi-$|\cdot|$-idealoids.

Introduction

In recent years a great deal of interest has been devoted to those (linear) projections π on a Banach space X satisfying either

$$\|x\|=\|\pi(x)\|+\|x-\pi(x)\|$$

or

$$\|x\|=\mathrm{Max}\{\|\pi(x)\|,\|x-\pi(x)\|\}.$$

In the former case π is called an L-projection on X and its range is an

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L-summand of X. In the latter case π is called an M-projection on X and its range is an M-summand of X. It is well known that the polar M^0 of an L-summand (resp. an M-summand) M of X is an M-summand (resp. and L-summand) of the dual space X'. Things do not go as well in the converse direction. Following Alfsen and Effros [1] an M-ideal of X is a closed subspace M of X such that M^0 is an L-summand of X' and it is well known that there are M-ideals which are not M-summands. At a first glance an analogous definition might be given for L-ideal, closed subspace M of X such that M^0 is an M-summand of X'. Surprisingly this extension of the concept of L-summand is trivial, that is L-ideals are L-summands [10; Theorem 1]. Up to date this striking asymmetry in the behaviour of the concepts of L- and M-summand seems to be an anecdotic fact whose last reason nobody has explored. This fact will find here a coherent explanation when considered in a wider context.

Behrends [3] considers L^p-summands for $1 < p < \infty$. An L^p-summand of X is the range of an L^p-projection on X that is a projection π on X satisfying

$$\|x\|^p = \|\pi(x)\|^p + \|x - \pi(x)\|^p$$

(L-summands and M-summands cover the cases $p = 1$ and $p = \infty$ respectively). A closed subspace M of X is an L^p-summand of X if and only if M^0 is an L^q-summand of X' where $1/p + 1/q = 1$ $(1 < p < \infty)$ [14; Proposition 2.9]. So when $p \neq \infty$ the concept of L^p-ideal agrees with the one of L^p-summand, and M-ideals become an even more isolated exception.

An extension of L-summands in a different direction is due to Lima [16] who introduced semi-L-summands. A definition of semi-L-summand which is equivalent to the one by Lima can be given as follows (see [16; Theorem 5.6]). By a semiprojection on X we mean a mapping π from X into X satisfying

$$\pi(x + \pi(y)) = \pi(x) + \pi(y) \quad \text{and} \quad \pi(\lambda x) = \lambda \pi(x)$$

for all x, y in X and scalar λ. A semi-L-summand of X is the range of a semi-L-projection on X, that is, a semiprojection π on X satisfying

$$\|x\| = \|\pi(x)\| + \|x - \pi(x)\|.$$

The existence of semi-L-summands which are not L-summands is known in [16]. Lima defines a semi-M-ideal as a closed subspace M of X such that M^0 is a semi-L-summand of X'. The class of semi-M-ideals is wider than the one of M-ideals. The concepts of semi-L^p-summand $(1 < p \leqq \infty)$ and semi-L^p-ideal $(1 \leqq p < \infty)$ can now be introduced in an analogous way but nobody has for the moment discussed them (Why?). Actually, as a consequence of the results in this paper semi-L^p-summands for $p \neq 1$ and semi-L^p-ideals for $p \neq \infty$ are in fact L^p-summands. So we realize that the existence of semi-L-summands which are not L-summands and of semi-M-ideals which are not M-ideals appears to be another surprising exception waiting for a coherent explanation.

The main purpose in this paper is to discuss the above-mentioned concepts in a wider and natural context. The above results are then extended as much as possible in our context. This gives a much more clarifying picture of the situation.

In a first step we consider those semiprojections π on a Banach space X with the property that the norm of any element x in X only depends on the norms of $\pi(x)$ and $x - \pi(x)$, that is, we suppose that there is a real function f defined in the first quadrant of $\mathbf{R}^2$ such that

$$\|x\| = f(\|\pi(x)\|, \|x - \pi(x)\|).$$

As noticed by Evans for linear π [12], when the trivial cases $\pi = 0, 1$ are excluded f must be the restriction to the first quadrant of a normalized absolute norm on $\mathbf{R}^2$ (in short: absolute norm) [8; Section 21]. Consequently, given an absolute norm $|\cdot|$ we define a semiprojection (resp. a projection) π on X to be a semi-$|\cdot|$-projection (resp. a $|\cdot|$-projection) if it satisfies

$$\|x\| = |(\|\pi(x)\|, \|x - \pi(x)\|)|.$$

Ranges of semi-$|\cdot|$-projections (resp. $|\cdot|$-projections) are called semi-$|\cdot|$-summands (resp. $|\cdot|$-summands). $|\cdot|$-Summands have been discussed in [21] and with different notation in [12, 19]. The nonlinear case has been considered in [20, 22].

As announced above for convenient selections of the norm $|\cdot|$ every semi-$|\cdot|$-summand is in fact a $|\cdot|$-summand. Our main result in section 1 reads as follows. Given an absolute norm $|\cdot|$ the following statements are equivalent:

(i) Every semi-$|\cdot|$-summand is a $|\cdot|$-summand.

(ii) $(1, 0)$ is not a vertex of the closed unit ball of the Banach space $(\mathbf{R}^2, |\cdot|)$.

The proof that (ii) $\Rightarrow$ (i) was first given in [20] and independently obtained in [22]. The fact that semi-M-summands are M-summands appears in the proof of the main result in [13]. The techniques used for the above theorem allow us to obtain also some interesting properties of semi-$|\cdot|$-projections as is, for example, the fact that they satisfy the Lipschitz condition

$$\|\pi(x) - \pi(y)\| \leqq \|x - y\|.$$

This has been proved by Yost for semi-L-projections [23]. By using the

Bishop–Phelps Theorem we prove that if a Banach space X is a semi-$|\cdot|$-summand of its bidual space X'' then X is a semi-L-summand of X''. This extends a result by Godefroy [15].

In a second step we discuss semi-$|\cdot|$-ideals. A semi-$|\cdot|$-ideal (resp. $|\cdot|$-ideal) of X is a closed subspace M of X whose polar M^0 is a semi-$|\cdot|^*$-summand (resp. $|\cdot|^*$-summand) of X' where $|\cdot|^*$ is defined by

$$|(r,s)|^* = \mathrm{Max}\{|rb+sa| : |(a,b)|=1\} \qquad (r,s\in\mathbf{R}).$$

The coherence of this definition is justified by the fact that every $|\cdot|$-summand is a $|\cdot|$-ideal. Our main result on semi-$|\cdot|$-ideals establishes that if $(0,1)$ is an extreme point in the closed unit ball of the Banach space $(\mathbf{R}^2, |\cdot|)$ then every semi-$|\cdot|$-ideal is a $|\cdot|$-summand. This improves the result in [21; Corollary 10]. We prove also that for each absolute norm $|\cdot|$not satisfying the above condition there are semi-$|\cdot|$-ideals which are not $|\cdot|$-ideals and $|\cdot|$-ideals which are not $|\cdot|$-summands. The second part of this assertion was announced in [21; Remark 12]. The relation between semi-$|\cdot|$-summands and semi-$|\cdot|$-ideals is also clarified by showing that if a closed subspace M of X is at the same time a semi-$|\cdot|$-summand and a semi-$|\cdot|$-ideal of X, then M is a $|\cdot|$-summand of X.

In the third section of this paper a new class of subspaces with no classical (L^p) counterpart appears. A closed subspace M of X is called a semi-$|\cdot|$-idealoid when M^0 is a semi-$|\cdot|^*$-ideal of X'. Every $|\cdot|$-ideal is clearly a semi-$|\cdot|$-idealoid, in fact M is a $|\cdot|$-ideal of X if and only if M is at the same time a semi-$|\cdot|$-ideal and a semi-$|\cdot|$-idealoid of X. If $(1,0)$ is not a vertex of the closed unit ball of $(\mathbf{R}^2, |\cdot|)$ then every semi-$|\cdot|$-idealoid is a $|\cdot|$-ideal. It was proved by Lima [16; Theorem 6.14] that M is a semi-L-summand of X if and only if M^0 is a semi-M-ideal of X'. We extend this result by showing that every semi-$|\cdot|$-summand is a semi-$|\cdot|$-idealoid and that if $(0,1)$ is an extreme point in the closed unit ball of $(\mathbf{R}^2, |\cdot|)$, then every semi-$|\cdot|$-idealoid is a semi-$|\cdot|$-summand. In view of the above facts the concept of semi-L^p-idealoid is not relevant, for semi-L-idealoids are semi-L-summands, semi-L^p-idealoids are L^p-summands whenever $1<p<\infty$ and semi-M-idealoids are M-ideals. However for any absolute norm $|\cdot|$ such that $(1,0)$ is a vertex and $(0,1)$ is not an extreme point of the closed unit ball of $(\mathbf{R}^2, |\cdot|)$ we construct a semi-$|\cdot|$-idealoid which is neither a semi-$|\cdot|$-summand nor a semi-$|\cdot|$-ideal.

It is a quite surprising fact that the way of consecutive predualizations of the concept of semisummand has its end at semiidealoids. More concretely we prove that every w^*-closed semiidealoid of a dual Banach space is a semisummand, so if M^0 is a semi-$|\cdot|^*$-idealoid of X' then M is a semi-$|\cdot|$-ideal of X. In particular we obtain that every w^*-closed ideal of a dual Banach space is a summand. This was asked by Alfsen and Effros [2; Problem 7.2] for M-ideals and affirmatively answered by Evans [13] and Lima [16].

Therefore, among the three concepts (semisummands, semiideals and semiidealoids) to be considered in our general context of absolute norms only two (summands and ideals) remain relevant in the linear case. Also only two of them (semisummands and semiideals) arise when we restrict our attention to classical L^p norms. We think that the appearance of semiidealoids adds new interest to the consideration of general absolute norms.

1. Semisummands

Throughout this paper $(X, \|\cdot\|)$ (or X if there is no ambiguity) will be a Banach space over the field $\mathbf{K}$ ($\mathbf{R}$ or $\mathbf{C}$). A mapping π from X into X will be called a *semiprojection* if it satisfies

$$\pi(x+\pi(y)) = \pi(x)+\pi(y) \qquad (x, y \in X), \tag{1.1}$$

$$\pi(\lambda x) = \lambda \pi(x) \qquad (\lambda \in \mathbf{K}, x \in X). \tag{1.2}$$

The range $\pi(X)$ of a semiprojection is a subspace of X but its kernel $\operatorname{Ker} \pi = \{x \in X : \pi(x) = 0\}$ is only a cone which remains invariant under scalar multiplication. The formula $x = \pi(x) + (x - \pi(x))$ gives the unique decomposition of any vector x as a sum of a vector in $\pi(X)$ and another in $\operatorname{Ker} \pi$. Note that a semiprojection π is a (linear) projection if and only if $\operatorname{Ker} \pi$ is convex.

Semiprojections appear in a very natural way. Suppose that M is a Chebyshev subspace of X, that is for every x in X there is a unique element $\pi(x)$ in M such that $\|x - \pi(x)\|$ is the distance from x to M. Then π (the best approximation mapping from X onto M) is a semiprojection.

By *absolute norm* we mean a norm $|\cdot|$ on $\mathbf{R}^2$ satisfying

$$|(r, s)| = |(|r|, |s|)| \qquad (r, s \in \mathbf{R}), \tag{1.3}$$

$$|(1,0)| = |(0,1)| = 1. \tag{1.4}$$

A number of geometric facts about absolute norms some of which will be needed in the sequel appear in [8; Section 21].

A semiprojection π on X will be called *absolute* if there is an absolute norm $|\cdot|$ such that

$$\|x\| = |(\|\pi(x)\|, \|x - \pi(x)\|)| \qquad (x \in X). \tag{1.5}$$

Every absolute semiprojection will be supposed nontrivial ($\pi \neq 0, 1$). Then the absolute norm which appears in (1.5) is clearly unique and the term *semi-*$|\cdot|$*-projection* will be applied to π in order to emphasize this norm. Ranges of absolute semiprojections will be called *semisummands* (or *semi-*$|\cdot|$*-summands*). If an absolute semiprojection is in fact linear we call it an *absolute projection* (or $|\cdot|$*-projection*) and its range will be a *summand* (or $|\cdot|$*-summand*).

The following intuitive classification of absolute norms will be useful in our study of absolute semiprojections and semisummands. We define the *type* of an absolute norm $|\cdot|$ to be 1 if $(1,0)$ is a vertex of the unit ball of $(\mathbf{R}^2, |\cdot|)$ (the unit ball has more than one support functional at $(1,0)$), 2 if $(1,0)$ is not a vertex but is an extreme point and ∞ if $(1,0)$ is not an extreme point. The *cotype* of $|\cdot|$ is defined in the same way but using $(0,1)$ instead of $(1,0)$. Equivalently, the cotype of the absolute norm $|\cdot|$ is the type of its (also absolute) reversed norm $|\cdot|^R$ defined by

$$(1.6) \qquad |(r,s)|^R = |(s,r)| \qquad (r,s \in \mathbf{R}).$$

When restricted to the first quadrant absolute norms are nondecreasing functions in each variable [8; Lemma 21.2]. This fact together with the following lemma which discusses the possibility of strict increasement will give the approximation properties of semisummands.

LEMMA 1.1. *Let $|\cdot|$ be an absolute norm and let r_1, r_2, s be nonnegative real numbers such that $r_1 < r_2$ and $|(r_1,s)| = |(r_2,s)|$. Then we have $|(r_1,s)| = |(r_2,s)| = s$ and the cotype of $|\cdot|$ is ∞. Conversely, suppose that the cotype of $|\cdot|$ is ∞. Then there is a positive real number r such that $|(r,1)| = 1$.*

PROOF. We can arrange $|(r_1,s)| = |(r_2,s)| = 1$ and we want $s = 1$. This follows from

$$1 = |(r_1,s)| = \left|\frac{r_1}{r_2}(r_2,s) + \left(1 - \frac{r_1}{r_2}\right)(0,s)\right| \leqq \frac{r_1}{r_2}\,|(r_2,s)| + \left(1-\frac{r_1}{r_2}\right)s$$

$$\leqq \frac{r_1}{r_2}\,|(r_2,s)| + \left(1-\frac{r_1}{r_2}\right)|(r_2,s)| = 1.$$

To obtain that the cotype of $|\cdot|$ is ∞ it is enough to write

$$(0,1) = \tfrac{1}{2}((r_2,1) + (-r_2,1))$$

which shows that $(0,1)$ is not an extreme point of the unit ball of $(\mathbf{R}^2, |\cdot|)$.

Conversely, if the cotype of $|\cdot|$ is ∞ we have

$$(0,1) = \tfrac{1}{2}((r,s_1) + (-r,s_2))$$

with $|(r, s_i)| = 1$ for $i = 1,2$, $(r, s_1) \neq (-r, s_2)$ and $r \geqq 0$. Then $|s_i| \leqq 1$ for $i = 1,2$ and $s_1 + s_2 = 2$, so $s_1 = s_2 = 1$ and $r > 0$.

If M is a closed subspace of the Banach space $(X, \|\cdot\|)$ we denote also by $\|\cdot\|$ the quotient norm, that is

$$\|x + M\| = \operatorname{Inf}\{\|x - m\| : m \in M\}.$$

For x in X we denote the set of best approximation of x in M by

$$P_M(x) = \{m \in M : \|x - m\| = \|x + M\|\}.$$

Recall that M is said to be *proximinal* when $P_M(x)$ is nonempty for all x in X. For m in M and $r \geqq 0$ we denote by $B_M(m, r)$ the closed ball in M with centre at m and radius r.

PROPOSITION 1.2. *Let π be a semi-$|\cdot|$-projection on X and $M = \pi(X)$. Let K be the greatest nonnegative real number such that $|(K, 1)| = 1$. Then*

$$P_M(x) = B_M(\pi(x), K\|x + M\|)$$

for all x in X. In particular M is a (closed) proximinal subspace of X. Also M is Chebyshev if and only if the cotype of $|\cdot|$ is 1 or 2.

PROOF. For $x \in X$ and $m \in M$ we have by the definition of semi-$|\cdot|$-projection that

$$\|x - m\| = |(\|\pi(x) - m\|, \|x - \pi(x)\|)| \geqq \|x - \pi(x)\|$$

so $\|x + M\| = \|x - \pi(x)\|$ for all x in X. Now assume without loss of generality that $\|x + M\| = 1$. Then

$$P_M(x) = \{m \in M : \|x - m\| = 1\} = \{m \in M : |(\|\pi(x) - m\|, 1)| = 1\}$$
$$= \{m \in M : \|\pi(x) - m\| \leqq K\}.$$

To conclude the proof note that $P_M(x)$ is a singleton for all x in X if and only if $K = 0$. By the above lemma this occurs if and only if the cotype of $|\cdot|$ is 1 or 2.

Consider the following questions. Does a semisummand M determine the absolute norm $|\cdot|$ for which M is a semi-$|\cdot|$-summand? If this is the case, does M determine the semi-$|\cdot|$-projection whose range is M? Both questions have an affirmative answer as we show below.

COROLLARY 1.3. *Let M be a semisummand of X. There is only one absolute semiprojection on X with range M.*

PROOF. Let π_1 and π_2 be absolute semiprojections on X with $\pi_1(X) = \pi_2(X) = M$. By the above proposition we have

$$P_M(x) = B_M(\pi_1(x), K_1 \| x + M \|) = B_M(\pi_2(x), K_2 \| x + M \|)$$

for all x in X and convenient constants K_1 and K_2. From the above equality we easily deduce that $\pi_1 = \pi_2$.

In accordance with earlier terminology we denote by L^p the classical absolute norm defined by

$$L^p(r,s) = (|r|^p + |s|^p)^{1/p} \qquad (r, s \in \mathbf{R}) \quad (1 \leqq p < \infty), \tag{1.7}$$

$$L^\infty(r,s) = \operatorname{Max}\{|r|, |s|\} \qquad (r, s \in \mathbf{R}) \quad (p = \infty). \tag{1.8}$$

We shall also denote the norm L^1 by L and the norm L^∞ by M. L^p-projections are of course the most relevant examples of absolute projection and they have been widely discussed (see for example [4], [5]). There is only one precedent of our semisummands, namely semi-L-summands introduced by Lima in [16]. By [16; Theorem 5.6] a semi-L-summand of X is a Chebyshev subspace of X whose best approximation mapping π satisfies (1.5) with $|\cdot| = L$. By Proposition 1.2 this is equivalent to our definition. Therefore semisummands generalize semi-L-summands in the same way that summands generalize L-summands. In particular a coherent definition of semi-L^p-summand for $p > 1$ arises. If semisummands were defined as Chebyshev subspaces whose best approximation mapping satisfies (1.5) for convenient absolute norm, then M-summands would not be semi-M-summands, in fact we should confine ourselves to absolute norms with cotype 1 or 2 according to the above proposition. This explains our approach in terms of semiprojections. As a consequence of the results in this section we shall obtain that semi-L^p-summands for $p > 1$ are in fact L^p-summands.

Behrends [3; Lemma 2.1] proves that if π_1 and π_2 are L^p-projections on a Banach space X for the same p and $\pi_i(X) = \pi_2(X)$ then $\pi_1 = \pi_2$. This had been proved before by Cunningham in case $p = 1$ [9; Lemma 2.1] and is known in [16] when π_1 and π_2 are semi-L-projections. All these results are very particular cases of the above corollary.

We now state the main result in this section.

THEOREM 1.4. *Let $|\cdot|$ be a fixed absolute norm. The following statements are equivalent.*

(i) *The type of $|\cdot|$ is 2 or ∞.*

(ii) *Every semi-$|\cdot|$-projection is in fact a $|\cdot|$-projection.*

Our proof of the above theorem involves some ideas on numerical ranges in Banach spaces. These techniques have been successfully applied in other directions (see [18] for example). More concretely, we shall obtain a formula relating the right-hand side derivatives at zero of the convex real functions $\alpha \to \|u + \alpha x\|$ and $\alpha \to \|u + \alpha\pi(x)\|$ where π is an absolute semiprojection on X, u is any norm-one element in $\pi(X)$ and $x \in X$ is arbitrary. We briefly recall the geometrical relevance of these derivatives.

For any norm-one element u in X we consider the state space of u, that is, the nonempty w^*-compact convex subset of the dual space X' defined by

$$D(u) = \{f \in X' : f(u) = \|f\| = 1\}. \tag{1.9}$$

For $x \in X$ we write

$$V(u, x) = \{f(x) : f \in D(u)\}; \tag{1.10}$$

$V(u, x)$ is a nonempty compact convex subset of the scalar field $\mathbf{K}$ and simple properties like

$$|\lambda| \leqq \|x\| \qquad \text{for all } \lambda \text{ in } V(u, x), \tag{1.11}$$

$$V(u, x + y) \subset V(u, x) + V(u, y) \qquad (x, y \in X), \tag{1.12}$$

$$V(u, \lambda u + \mu x) = \lambda + \mu V(u, x) \qquad (\lambda, \mu \in \mathbf{K}) \tag{1.13}$$

are easily verified. We shall also write

$$v(u, x) = \operatorname{Max}\{|\lambda| : \lambda \in V(u, x)\} \tag{1.14}$$

and it follows from the above statements that $v(u, \cdot)$ is a seminorm on X satisfying

$$v(u, x) \leqq \|x\| \qquad \text{for all } x \text{ in } X. \tag{1.15}$$

It is well known (see [11; Theorem V.9.5] for example) that

$$\operatorname{Max} \operatorname{Re} V(u, x) = \lim_{\substack{\alpha \to 0 \\ \alpha > 0}} \frac{1}{\alpha}(\|u + \alpha x\| - 1). \tag{1.16}$$

The following elementary lemma deals with the case in which X is $\mathbf{R}^2$ provided with an absolute norm. We define the *numerical index* $n(|\cdot|)$ of the absolute norm $|\cdot|$ by

$$n(|\cdot|) = \lim_{\substack{\alpha \to 0 \\ \alpha > 0}} \frac{1}{\alpha}(|(1, \alpha)| - 1) = \operatorname{Max} V((1,0), (0,1)). \tag{1.17}$$

It is clear that $0 \leqq n(|\cdot|) \leqq 1$. Also $n(L)=1$ and $n(L^p)=0$ for $1<p\leqq\infty$.

LEMMA 1.5. *Let* $\mathbf{R}^2$ *be provided with an absolute norm* $|\cdot|$. *Then*
(i) $V((1,0),(r,s))=[r-n(|\cdot|)|s|, r+n(|\cdot|)|s|]$ *for all* (r,s) *in* $\mathbf{R}^2$.
(ii) $|(r,s)| \geqq |r|+n(|\cdot|)|s|$ *for all* (r,s) *in* $\mathbf{R}^2$.
(iii) *The type of* $|\cdot|$ *is* 1 *if and only if* $n(|\cdot|)>0$.

PROOF. Part (i) follows almost inmediately from the definition of the numerical index and (1.13). From (i) we deduce

$$v((1,0),(r,s)) = |r| + n(|\cdot|)|s|$$

and then (ii) follows from (1.15). For (iii) note that (1,0) is a vertex of the unit ball of $\mathbf{R}^2$ if and only if $D((1,0))$ separates the points in $\mathbf{R}^2$ which turns out to be equivalent to the fact that $v((1,0),\cdot)$ is a norm on $\mathbf{R}^2$, but in view of the above formula for $v((1,0),\cdot)$ this occurs if and only if $n(|\cdot|)>0$.

Our next Lemma is an elementary extension of (1.16).

LEMMA 1.6. *Let F be a function from a real interval* $[0,\delta]$ *into X. Assume that* $\|F(0)\|=1$ *and that F is differentiable at zero. Then the real function G defined in* $[0,\delta]$ *by* $G(\alpha)=\|F(\alpha)\|$ *is differentiable at zero and*

$$G'(0) = \operatorname{Max} \operatorname{Re} V(F(0), F'(0)).$$

PROOF. Write $F(\alpha)=F(0)+\alpha F'(0)+\alpha H(\alpha)$ where $\lim_{\alpha\to 0}\|H(\alpha)\|=0$. Then we have

$$\frac{1}{\alpha}(\|F(0)+\alpha F'(0)\|-1)-\|H(\alpha)\| \leqq \frac{1}{\alpha}(G(\alpha)-1)$$
$$\leqq \frac{1}{\alpha}(\|F(0)+\alpha F'(0)\|-1)+\|H(\alpha)\|$$

and the result follows from (1.16) by letting $\alpha\to 0$.

We go now towards the crucial result in this section. For any nonzero subspace M of X we can define a seminorm ρ_M on X by

(1.18) $$\rho_M(x) = \operatorname{Sup}\{v(u,x) : u \in M, \|u\|=1\}.$$

It is clear that

(1.19) $$\rho_M(x) \leqq \|x\| \qquad \text{for all } x \text{ in } X,$$

(1.20) $$\rho_M(m) = \|m\| \qquad \text{for all } m \text{ in } M.$$

For $\lambda \in \mathbf{K}$ and $r \geqq 0$ we write $E(\lambda, r) = \{\mu \in \mathbf{K} : |\mu - \lambda| \leqq r\}$.

THEOREM 1.7. *Let π be a semi-$|\cdot|$-projection on X and $M = \pi(X)$. For m in M with $\|m\| = 1$ and x in X we have*
(i) $V(m, x) = V(m, \pi(x)) + E(0, n(|\cdot|)\|x - \pi(x)\|)$,
(ii) $v(m, x) = v(m, \pi(x)) + n(|\cdot|)\|x - \pi(x)\|$,
(iii) $\rho_M(x) = \|\pi(x)\| + n(|\cdot|)\|x - \pi(x)\|$.

PROOF. By (1.16) and the definition of semi-$|\cdot|$-projection we have

$$\operatorname{Max} \operatorname{Re} V(m, x) = \lim_{\substack{\alpha \to 0 \\ \alpha > 0}} \frac{1}{\alpha}(|(\|m + \alpha\pi(x)\|, \alpha\|x - \pi(x)\|)| - 1).$$

Consider the function $F : [0, 1] \to \mathbf{R}^2$ defined by

$$F(\alpha) = (\|m + \alpha\pi(x)\|, \alpha\|x - \pi(x)\|) \qquad (0 \leqq \alpha \leqq 1).$$

We use again (1.16) to see that F is defferentiable at zero with

$$F'(0) = (\operatorname{Max} \operatorname{Re} V(m, \pi(x)), \|x - \pi(x)\|)$$

while it is clear that $F(0) = (1, 0)$. Now we apply consecutively Lemmas 1.6 and 1.5 to obtain

$$\begin{aligned} \operatorname{Max} \operatorname{Re} V(m, x) &= \operatorname{Max} \operatorname{Re} V((1, 0), F'(0)) \\ &= \operatorname{Max} \operatorname{Re} V(m, \pi(x)) + n(|\cdot|)\|x - \pi(x)\| \\ &= \operatorname{Max} \operatorname{Re}(V(m, \pi(x)) + E(0, n(|\cdot|)\|x - \pi(x)\|)). \end{aligned}$$

Change x by λx with $\lambda \in \mathbf{K}$, $|\lambda| = 1$ to obtain that the compact convex subsets of $\mathbf{K}$ appearing in (i) have the same support mapping, so they agree (see [6; p. 90]).
Part (ii) is a direct consequence of (i) and then (iii) follows from (ii) and (1.20).

The following Corollary is known (see [20; Theorem 10.6], [22]) and it extends the particular case in which the absolute norm under consideration is M (see the proof of [13; Theorem]). With this Corollary we prove the assertion (i) $\Rightarrow$ (ii) in our Theorem 1.4.

COROLLARY 1.8. *Let π be an absolute semiprojection on X. If the type of the absolute norm associated to π is 2 or ∞, then π is linear, hence an absolute projection on X.*

PROOF. If the type of the absolute norm $|\cdot|$ is not 1 we have by the last part of Lemma 1.5 that $n(|\cdot|) = 0$ and Theorem 1.7 gives $\rho_M(x) = \|\pi(x)\|$ for all x in X

where $M = \pi(X)$. So we have that

$$\operatorname{Ker} \pi = \{x \in X : \rho_M(x) = 0\}$$

which is a subspace of X.

As noticed before the above Corollary implies that semi-L^p-summands for $1 < p < \infty$ are L^p-summands. This together with Proposition 1.2 gives the following characterization of L^p-summands.

COROLLARY 1.9. *Given a subspace M of X and a fixed p with $1 < p < \infty$, the following statements are equivalent.*

(i) *M is an L^p-summand of X,*

(ii) *M is Chebyshev and the best approximation mapping π from X onto M satisfies*

$$\| x \|^p = \| \pi(x) \|^p + \| x - \pi(x) \|^p$$

for all x in X.

Up to this moment no concrete example of semisummand has been given. Of course classical L^p-spaces are rich in L^p-summands. Without leaving the context of classical Banach spaces the following could be an elementary example of $|\cdot|$-summand for nonclassical absolute norm $|\cdot|$.

For each real number γ with $0 < \gamma < 1$ consider the absolute norm $|\cdot|_\gamma$ defined by

$$|(r, s)|_\gamma = \operatorname{Max}\{|s|, |r| + \gamma |s|\}.$$

Let X_γ be the subspace of l_3^∞ defined by the equation

$$x_2 - x_3 = 2\gamma x_1.$$

It is not difficult to verify that $\mathbf{R}(0, 1, 1,)$ is a $|\cdot|_\gamma$-summand of X_γ. What about semi-$|\cdot|$-summands which are not $|\cdot|$-summands? The easiest example of a semi-L-summand which is not an L-summand is $\mathbf{R}(1, 1, 1)$ in l_3^∞. With a bit of additional effort and without leaving the frame of classical Banach spaces we can show some elementary examples of semi-$|\cdot|$-summands which are not $|\cdot|$-summands for nonclassical absolute norms $|\cdot|$. Concretely, for each γ with $0 < \gamma < 1$ there is a subspace X_γ of l_6^∞ such that $\mathbf{R}(1, 1, 1, 0, 0, 0)$ is a semi-$|\cdot|_\gamma$-summand but is not a $|\cdot|_\gamma$-summand of X_γ. Just take for X_γ the subspace of l_6^∞ defined by the equations

$$2\gamma x_4 - x_1 + x_2 = 2\gamma x_5 - x_2 + x_3 = 2\gamma x_6 - x_3 + x_1 = 0.$$

The following is a straightforward consequence of Corollary 1.8, Lemma 1.5(iii) and Theorem 1.7(iii). If π is an absolute semiprojection on X which is not linear and $M = \pi(X)$, then ρ_M is an equivalent norm on X and π becomes a semi-L-projection on the Banach space (X, ρ_M). So we can only get semisummands which are not summands by renorming Banach spaces with semi-L-summands which are not L-summands. Our next theorem shows that this renorming process can always be carried over. We need the following property of absolute norms.

LEMMA 1.10. *Let $|\cdot|$ be an absolute norm and write $n = n(|\cdot|)$. There is a unique absolute norm $|\cdot|^+$ such that*

$$|(a,b)| = |(|a| + n|b|, |b|)|^+$$

for all a, b in $\mathbf{R}$.

PROOF. Define a continuous real function ϕ on $[0,1]$ by

$$\phi(t) = |(1-(n+1)t, t)| \text{ for } 0 \leqq t \leqq \frac{1}{n+1}, \quad \phi(t) = t \text{ for } \frac{1}{n+1} \leqq t \leqq 1.$$

The restrictions of ϕ to the intervals $[0, 1/(n+1)]$ and $[1/(n+1), 1]$ are convex functions, so to prove that ϕ is convex we take $0 \leqq t_1 < 1/(n+1) < t_2 \leqq 1$ and we must verify that

$$\phi\left(\frac{t_1+t_2}{2}\right) \leqq \frac{1}{2}\phi(t_1) + \frac{1}{2}\phi(t_2).$$

This is clear when $(t_1+t_2)/2 \geqq 1/(n+1)$. Otherwise we have

$$\phi\left(\frac{t_1+t_2}{2}\right) = \left|\left(1-(n+1)\frac{t_1+t_2}{2}, \frac{t_1}{2}\right) + \left(0, \frac{t_2}{2}\right)\right|$$

$$\leqq \frac{1}{2}|(2-(n+1)(t_1+t_2), t_1)| + \frac{1}{2}t_2 \leqq \frac{1}{2}|(1-(n+1)t_1, t_1)| + \frac{1}{2}t_2$$

where the last inequality follows from

$$0 \leqq 2-(n+1)(t_1+t_2) \leqq 1-(n+1)t_1.$$

We have clearly $\phi(t) \geqq t$ for $0 \leqq t \leqq 1$. By Lemma 1.5(ii) we have also $\phi(t) \geqq 1-t$ for $0 \leqq t \leqq 1/(n+1)$ and the inequality is clear when $1/(n+1) \leqq t \leqq 1$. So ϕ satisfies the condition

$$\operatorname{Max}\{1-t, t\} \leqq \phi(t) \leqq 1 \qquad (0 \leqq t \leqq 1).$$

By [8; Lemma 21.3] there is a unique absolute norm $|\cdot|^+$ such that

$$|(1-t,t)|^+=\phi(t) \qquad (0\leqq t\leqq 1).$$

To prove that $|\cdot|^+$ satisfies the required condition it is enough to apply its defining formula with

$$t=\frac{|b|}{|a|+(n+1)|b|}\leqq\frac{1}{n+1}$$

where $a,\ b\in\mathbf{R}$, $|a|+|b|>0$. The uniqueness of $|\cdot|^+$ is easy and will not be needed below, so we leave it as an exercise.

THEOREM 1.11. *Let* ${}_0|\cdot|$ *and* $|\cdot|$ *be arbitrary type* 1 *absolute norms,* π *a semi-*${}_0|\cdot|$*-projection on* X *and* $M=\pi(X)$. *Define*

$$|||x|||=\left|\left(\rho_M(x),\frac{n_0}{n}\|x+M\|\right)\right|^+ \qquad (x\in X)$$

where $n_0=n({}_0|\cdot|)$ *and* $n=n(|\cdot|)$. *Then* $|||\cdot|||$ *is an equivalent norm on* X *and* π *becomes a semi-*$|\cdot|$*-projection on the Banach space* $(X,|||\cdot|||)$. *When* ${}_0|\cdot|=L$ *we get*

$$|||x|||=\left|\left(\|x\|,\frac{1}{n}\|x+M\|\right)\right|^+ \qquad (x\in X)$$

while if $|\cdot|=L$ *we have*

$$|||x|||=\rho_M(x) \qquad (x\in X).$$

PROOF. It is clear that $|||\cdot|||$ is a seminorm on X satisfying

$$|||x|||\leqq(1+n_0/n)\|x\| \qquad \text{for all } x \text{ in } X.$$

By Theorem 1.7(iii) we have also

$$\|x\|\leqq(1/n_0)\rho_M(x)\leqq(1/n_0)|||x||| \qquad \text{for all } x \text{ in } X,$$

so $|||\cdot|||$ is an equivalent norm on X.

Another application of Theorem 1.7(iii) and the definition of $|\cdot|^+$ given in the above Lemma give us

$$|||x|||=\left|\left(\|\pi(x)\|+n_0\|x+M\|,\frac{n_0}{n}\|x+M\|\right)\right|^+=\left|\left(\|\pi(x)\|,\frac{n_0}{n}\|x+M\|\right)\right|.$$

Now we put $\pi(x)$ and $x-\pi(x)$ instead of x in the above equality to obtain

$$|||\pi(x)|||=\|\pi(x)\| \quad\text{and}\quad |||x-\pi(x)|||=\frac{n_0}{n}\|x+M\|$$

so that the equality reads

$$|||x||| = |(|||\pi(x)|||, |||x - \pi(x)|||)|$$

and we have proved that π is a semi-$|\cdot|$-projection on $(X, |||\cdot|||)$. In the particular case ${}_0|\cdot| = L$ we have $n_0 = 1$ and $\rho_M = \|\cdot\|$. On the other hand if $|\cdot| = L$ it is easy to verify that $|\cdot|^+ = M$ and $n = 1$, so

$$|||x||| = \mathrm{Max}\{\rho_M(x), n_0\|x + M\|\} = \rho_M(x) \qquad \text{for all } x \text{ in } X.$$

REMARK 1.12. Let $(X, |||\cdot|||)$ be the Banach space resulting from the above Theorem. If we apply to $(X, |||\cdot|||)$ the same Theorem interchanging ${}_0|\cdot|$ and $|\cdot|$ we get again the initial Banach space $(X, \|\cdot\|)$.

Now the proof of Theorem 1.4 has been concluded. By Corollary 1.8 we can only have semi-$|\cdot|$-summands which are not $|\cdot|$-summands when the type of the absolute norm $|\cdot|$ is 1. Conversely, given a type 1 absolute norm $|\cdot|$ we can construct a semi-$|\cdot|$-summand which is not a $|\cdot|$-summand, for it is enough to apply the renorming process of the last Theorem to a space with a semi-L-summand which is not an L-summand. All known examples of semi-L-summands which are not L-summands appear in real Banach spaces. It is an open question whether or not every semi-L-summand in a complex Banach space is in fact an L-summand (see [25]). In view of the above comments the analogous question for general semisummands reduces to semi-L-summands. This is an example showing that Theorem 1.11 reduces up to a point the study of semisummands to the one of semi-L-summands. Next we give another application of this idea.

It is clear that every absolute projection π on X is continuous with norm one, so it satisfies

$$\|\pi(x) - \pi(y)\| \leqq \|x - y\| \qquad \text{for all } x, y \text{ in } X.$$

It has been proved by Yost [23; Theorem 1.3] that semi-L-projections satisfy the above inequality. We now extend this result to absolute semiprojections.

COROLLARY 1.13. *Let π be an absolute semiprojection on X. Then*

$$\|\pi(x) - \pi(y)\| \leqq \|x - y\| \qquad \textit{for all } x, y \textit{ in } X.$$

PROOF. By Corollary 1.8 and the above comments we can suppose that the type of the absolute norm associated to π is 1. Then by Theorem 1.11 we have that π is a semi-L-projection of the Banach space (X, ρ_M) where $M = \pi(X)$. By applying (1.20), [23; Theorem 1.3] and (1.19) we obtain

$$\|\pi(x)-\pi(y)\|=\rho_M(\pi(x)-\pi(y))\leqq\rho_M(x-y)\leqq\|x-y\|.$$

Let X be complex and denote by $X_{\mathbf{R}}$ the underlying real Banach space. Every semisummand of X is clearly a semisummand of $X_{\mathbf{R}}$. The converse is not true (**R** is an L^2-summand of $\mathbf{C}_{\mathbf{R}}$). It was proved in [21; Corollary 7] that when the euclidean norm L^2 is excluded every summand of $X_{\mathbf{R}}$ is a summand of X. This was obtained as a consequence of the fact that for $|\cdot|\neq L^2$ every $|\cdot|$-summand in a real or complex Banach space is invariant under any bounded linear operator whose algebra numerical range has empty interior in **K** [21; Theorem 6]. In fact Partington [19] had proved the complex version of this theorem in a more general form. Next we obtain the extension to semisummands of that result.

COROLLARY 1.14. *Let M be a semi-$|\cdot|$-summand of X and T a bounded linear operator on X. If $|\cdot|\neq L^2$ and the algebra numerical range of T has empty interior, then $T(M)\subset M$.*

PROOF. By Corollary 1.8 and [21; Theorem 6] we can suppose that the type of $|\cdot|$ is 1, that is, $n(|\cdot|)>0$. Let π be the semi-$|\cdot|$-projection on X with range M and let us fix m in M with $\|m\|=1$. The first part of Theorem 1.7 for $x=T(m)$ gives

$$V(m,T(m))=V(m,\pi T(m))+E(0,n(|\cdot|)\|T(m)-\pi T(m)\|).$$

Since $V(m,T(m))$ is included in the algebra numerical range of T (see [7; p. 82]) we have that $V(m,T(m))$ has empty interior. This implies $\pi T(m)=T(m)$ as required.

COROLLARY 1.15. *Let X be complex. The semisummands of X are the same as the ones of $X_{\mathbf{R}}$, provided that the euclidean norm L^2 is excluded.*

PROOF. Analogous to the one of [21; Corollary 7] by using the above Corollary instead of [21; Theorem 6]. In this way one obtains that every semisummand M of $X_{\mathbf{R}}$ is a (complex) subspace of X. This and Proposition 1.2 give that the absolute semiprojection from $X_{\mathbf{R}}$ onto M is a (complex) semiprojection on X.

Godefroy [15; Theorem 6] has proved that if a Banach space is a summand of its bidual space, then it is an L-summand. We conclude this section with an independent proof of this result and at the same time we consider the nonlinear case. We denote by J_X the canonical imbedding of a Banach space X into its bidual space X''.

LEMMA 1.16. *$\rho_{J_X(X)}(F) = \|F\|$ for all F in X''.*

PROOF. For x in X with $\|x\| = 1$ and f in $D(X, x)$ we have that $J_{X'}(f)$ belongs to $D(X'', J_X(x))$. Now apply the Bishop–Phelps Theorem (see [8; §16]) to obtain that $\rho_{J_X(X)}(F) \geqq \|F\|$ for all F in X''.

THEOREM 1.17. *Assume that $J_X(X)$ is a semisummand of X''. Then $J_X(X)$ is a semi-L-summand of X''.*

PROOF. Use the above Lemma and Theorem 1.7(iii).

2. Semiideals

Together with an absolute norm $|\cdot|$ we can consider another one $|\cdot|^*$ defined by

$$|(r,s)|^* = \operatorname{Sup}\{|rb + sa| : |(a,b)| = 1\}. \tag{2.1}$$

It is easy to verify that if M is a $|\cdot|$-summand of X, then its polar

$$M^0 = \{f \in X' : f(m) = 0 \text{ for all } m \text{ in } M\}$$

is a $|\cdot|^*$-summand of the dual space X'. In [21] the $|\cdot|$-*ideals* of X were defined as those closed subspaces M of X such that M^0 is a $|\cdot|^*$-summand of X', so that every $|\cdot|$-summand of X is a $|\cdot|$-ideal of X. The converse is not true when $|\cdot| = M$, that is, there are M-ideals which are not M-summands (see [1]). Lima [16] defines a semi-M-ideal of X to be a closed subspace M of X such that M^0 is a semi-L-summand of X'. The concept of semiideal which we now introduce includes both $|\cdot|$-ideals and semi-M-ideals.

A closed subspace M of X will be called a *semi-$|\cdot|$-ideal* of X if M^0 is a semi-$|\cdot|^*$-summand of X'. We say that M is a *semiideal* (respectively an *ideal*) of X if there is an absolute norm $|\cdot|$ such that M is a semi-$|\cdot|$-ideal (respectively a $|\cdot|$-ideal) of X. In view of Corollary 1.3 the absolute norm $|\cdot|$ is unique.

It is clear that every $|\cdot|$-ideal (hence every $|\cdot|$-summand) of X is a semi-$|\cdot|$-ideal of X. However semi-$|\cdot|$-summands need not be semi-$|\cdot|$-ideals. In fact let X be a Banach space with a semi-L-summand M which is not an L-summand. If M were a semi-L-ideal of X then M^0 would be a semi-M-summand of X', that is (Corollary 1.8) an M-summand of X'. Then by [10; Theorem 1] M would be an L-summand of X, a contradiction. Thus semisummands and semiideals are different generalizations of summands. The relation between summands and semiideals will be completely clarified in this section just as the relation of summands with semisummands was clarified in section 1. The first step will be to prove that under certain condition on the absolute norm $|\cdot|$ every semi-$|\cdot|$-ideal is in fact a $|\cdot|$-summand.

LEMMA 2.1. *Let $|\cdot|$ be an absolute norm. The type of $|\cdot|^*$ is 1 if and only if the cotype of $|\cdot|$ is ∞.*

PROOF. Each element (a, b) in $(\mathbf{R}^2, |\cdot|)$ can be identified with the continuous linear functional on $(\mathbf{R}^2, |\cdot|^*)$ given by $(r, s) \to rb + sa$. With this identification in mind the states of $(1,0)$ in $(\mathbf{R}^2, |\cdot|^*)$ are those elements in $\mathbf{R}^2$ of the form $(x, 1)$ with $|(x, 1)| = 1$. So the type of $|\cdot|^*$ is 1 if and only if there is a $x \neq 0$ such that $|(x, 1)| = 1$. By Lemma 1.1 this occurs if and only if the cotype of $|\cdot|$ is ∞.

From the above Lemma the relation between the type of $|\cdot|^*$ and the cotype of $|\cdot|$ can be completely clarified. Suggestively, if the type of $|\cdot|^*$ is p and the cotype of $|\cdot|$ is q, then $1/p + 1/q = 1$ with the usual conventions.

THEOREM 2.2. *Assume that the cotype of the absolute norm $|\cdot|$ is not ∞. Then every semi-$|\cdot|$-ideal is a $|\cdot|$-summand.*

PROOF. If M is a semi-$|\cdot|$-ideal of X, then M^0 is a semi-$|\cdot|^*$-summand of X'. By the above Lemma the type of $|\cdot|^*$ is not 1, so by Corollary 1.8 M^0 is a $|\cdot|^*$-summand of X' and M is a $|\cdot|$-ideal of X. By [21; Corollary 10] M is a $|\cdot|$-summand of X.

The above Theorem improves [21; Corollary 10] where under the same assumption on the norm $|\cdot|$ it was obtained that every $|\cdot|$-ideal is a $|\cdot|$-summand. Our Theorem implies that semi-L^p-ideals for $1 \leqq p < \infty$ are L^p-summands so improving the results in [10; Theorem 1] and [14; Proposition 2.9]. The assumption that the cotype of $|\cdot|$ is not ∞ can not be dropped, for there are semi-M-ideals which are not M-ideals (even less M-summands) and M-ideals which are not M-summands. Next we give an example of semi-$|\cdot|$-ideal for nonclassical absolute norm $|\cdot|$. For each real number γ satisfying $0 < \gamma < 1$ let X_γ be the subspace of l_4^1 defined by the equation

$$\gamma(x_1 + x_2 + x_3) = (1 - \gamma)x_4$$

and let M be defined by

$$x_1 + x_2 + x_3 = x_4 = 0.$$

It is not difficult to verify that M is a semi-$|\cdot|_\gamma$-ideal of X_γ for all γ, the absolute norm $|\cdot|_\gamma$ being defined by

$$|(r, s)|_\gamma = \mathrm{Max}\{|s|, |r| + \gamma|s|\} \qquad (r, s \in \mathbf{R}).$$

Moreover, M is not a $|\cdot|_\gamma$-ideal of X_γ. Our next Theorem will show that the assumption on the norm $|\cdot|$ in Theorem 2.2 is in fact necessary. More concretely, for each cotype ∞ absolute norm $|\cdot|$ there are semi-$|\cdot|$-ideals which are not $|\cdot|$-ideals and $|\cdot|$-ideals which are not $|\cdot|$-summands.

THEOREM 2.3. *Let M be a semi-M-ideal of $(X, \|\cdot\|)$ and $|\cdot|$ a cotype ∞ absolute norm. Define*

$$|||x||| = |(n(|\cdot|^*)\|x\|, \|x+M\|)| \qquad (x \in X).$$

Then $|||\cdot|||$ is an equivalent norm on X and M is a semi-$|\cdot|$-ideal of $(X, |||\cdot|||)$. Also M is a $|\cdot|$-ideal of $(X, |||\cdot|||)$ if and only if M is an M-ideal of $(X, \|\cdot\|)$ and M is a $|\cdot|$-summand of $(X, |||\cdot|||)$ if and only if M is an M-summand of $(X, \|\cdot\|)$.

PROOF. By Lemma 2.1 the type of $|\cdot|^*$ is 1, that is (Lemma 1.5) $n(|\cdot|^*) > 0$. So if we define

$$\|(x, y+M)\| = |(n(|\cdot|^*)\|x\|, \|y+M\|)| \qquad (x, y \in X)$$

we obtain a norm on the product space $Y = X \times (X/M)$. The mapping ϕ defined by

$$\phi(x) = (x, x+M) \qquad (x \in X)$$

is linear and injective, so $|||\cdot|||$ is a norm on X with which ϕ is isometric. The straightforward inequalities

$$n(|\cdot|^*)\|x\| \leqq |||x||| \leqq (1 + n(|\cdot|^*))\|x\|$$

show that $|||\cdot|||$ is equivalent to $\|\cdot\|$.

Now we consider the natural identification of the dual space Y' of Y with the product space $X' \times M^0$. Since $X \times \{0\}$ is clearly a $|\cdot|$-summand of Y we deduce that the norm of Y' is then given by

$$\text{(a)} \qquad \|(f, g)\| = \left|\left(\|g\|, \frac{1}{n(|\cdot|^*)}\|f\|\right)\right|^* \qquad (f \in X', g \in M^0).$$

(Observe that we use the same notation for each norm and its dual one.)

Consider ϕ as an isometric linear bijection from $(X, |||\cdot|||)$ onto $\phi(X)$. Then the transpose mapping ϕ^t is an isometric linear bijection from $\phi(X)'$ onto $(X', |||\cdot|||)$. Also $\phi(X)'$ can be identified with the quotient space $Y'/\phi(X)^0 \equiv (X' \times M^0)/\phi(X)^0$. This gives that the mapping $\psi : (X' \times M^0)/\phi(X)^0 \to X'$ defined by

$$\psi((f, g) + \phi(X)^0) = f + g \qquad (f \in X', g \in M^0)$$

is an isometry. Since we have easily $\phi(X)^0 = \{(g, -g) : g \in M^0\}$ we obtain in view of (a) the following formula for the norm $|||\cdot|||$ on X':

$$\text{(b)} \qquad |||f||| = |||\psi((f,0)+\phi(X)^0)||| = \|(f,0)+\phi(X)^0\|$$

$$= \operatorname{Inf}\{\|(f+g,-g)\| : g \in M^0\} = \operatorname{Inf}\left\{\left|\left(\|g\|, \frac{1}{n(|\cdot|^*)}\|f+g\|\right)\right|^* : g \in M^0\right\}.$$

Recall that M is a semi-M-ideal of $(X, \|\cdot\|)$, so let π denote the semi-L-projection from $(X', \|\cdot\|)$ onto M^0. Then we have

$$\|f+g\| = \|\pi(f)+g\| + \|f-\pi(f)\|$$

for all f in X' and g in M^0. In this way (b) reads

$$\text{(c)} \qquad |||f||| = \operatorname{Inf}\left\{\left|\left(\|g\|, \frac{1}{n(|\cdot|^*)}(\|\pi(f)+g\| + \|f-\pi(f)\|)\right)\right|^* : g \in M^0\right\}.$$

In particular (choose $g = -\pi(f)$),

$$\text{(d)} \qquad |||f||| \leqq \left|\left(\|\pi(f)\|, \frac{1}{n(|\cdot|^*)}\|f-\pi(f)\|\right)\right|^*.$$

We shall prove that this inequality is actually an equality. To this end we fix g in M^0 and apply Lemma 1.10 to the absolute norm $|\cdot|^*$ with

$$a = \|g\| \quad \text{and} \quad b = \frac{1}{n(|\cdot|^*)}(\|\pi(f)+g\| + \|f-\pi(f)\|)$$

and we obtain

$$\left|\left(\|g\|, \frac{1}{n(|\cdot|^*)}(\|\pi(f)+g\| + \|f-\pi(f)\|)\right)\right|^* = |(a,b)|^* = |(a+n(|\cdot|^*)b, b)|^{*+}$$

$$= \left|\left(\|g\| + \|\pi(f)+g\| + \|f-\pi(f)\|, \frac{1}{n|\cdot|^*)}(\|\pi(f)+g\| + \|f-\pi(f)\|)\right)\right|^{*+}$$

$$\geqq \left|\left(\|\pi(f)\| + \|f-\pi(f)\|, \frac{1}{n(|\cdot|^*)}\|f-\pi(f)\|\right)\right|^{*+}$$

$$= \left|\left(\|\pi(f)\|, \frac{1}{n(|\cdot|^*)}\|f-\pi(f)\|\right)\right|^*.$$

Thus we have proved that

$$\text{(2.3)} \qquad |||f||| = \left|\left(\|\pi(f)\|, \frac{1}{n(|\cdot|^*)}\|f-\pi(f)\|\right)\right|^*.$$

It is enough to put in the above formula $\pi(f)$ and $f-\pi(f)$ instead of f to obtain

$$|||f|||=|(|||\pi(f)|||, |||f-\pi(f)|||)|^*.$$

This shows that π is a semi-$|\cdot|^*$-projection on $(X', |||\cdot|||)$, so M^0 is a semi-$|\cdot|^*$-summand of $(X', |||\cdot|||)$, that is, M is a semi-$|\cdot|$-ideal of $(X, |||\cdot|||)$.

Note that the absolute semiprojection onto M^0 remains unchanged in the above renorming process, so M is a $|\cdot|$-ideal of $(X, |||\cdot|||)$ if and only if it is an M-ideal of $(X, \|\cdot\|)$. It is almost clear that if M is an M-summand of $(X, \|\cdot\|)$ then M is a $|\cdot|$-summand of $(X, |||\cdot|||)$. Assume to conclude that M is a $|\cdot|$-summand of $(X, |||\cdot|||)$ and let P be the corresponding $|\cdot|$-projection. Then $1-P'$ is a $|\cdot|^*$-projection on $(X', |||\cdot|||)$ with range M^0, so we have by Corollary 1.3 that $1-P'=\pi$ and this implies that π is linear and is an L-projection on $(X', \|\cdot\|)$, hence P is an M-projection on $(X, \|\cdot\|)$ and M is an M-summand of $(X, \|\cdot\|)$.

REMARK 2.4. The formula (2.3) which appears in the proof of the above Theorem agrees with the one given by Theorem 1.11 in order to turn the semi-L-projection π into a semi-$|\cdot|^*$-projection. So we have proved that when the renorming process in Theorem 1.11 is applied to a dual Banach space with a w^*-closed semi-L-summand, then the new norm is again dual.

Our next goal will be to prove that the renorming process in the above theorem is reversible so that every semi-$|\cdot|$-ideal must arise by renorming a space with a semi-M-ideal as done in the Theorem. We need the following result on absolute semiprojections on dual Banach spaces.

LEMMA 2.5. *Let π be a semi-$|\cdot|$-projection on a dual Banach space X. Let $\{y_\alpha\}$ be a net satisfying $\|y_\alpha\|\leqq 1$ and $y_\alpha\in \operatorname{Ker}\pi$ for all α and suppose that $\{y_\alpha\}$ converges in the w^*-topology to $x+y$ with x in $\pi(X)$ and y in* Ker π. *Then we have*

$$\|x\|+n(|\cdot|)\|y\|\leqq n(|\cdot|).$$

PROOF. For arbitrary $t>0$ the net $\{t\|x\|y_\alpha+x\}$ converges in the w^*-topology to $(1+t\|x\|)x+t\|x\|y$. So from

$$\|t\|x\|y_\alpha+x\|=|(\|x\|, t\|x\|\|y_\alpha\|)|\leqq\|x\||(1,t)|$$

and the w^*-lower semicontinuity of the norm of X we deduce

$$\|(1+t\|x\|)x+t\|x\|y\|\leqq\|x\||(1,t)|.$$

On the other hand we have by Lemma 1.5 that

$$\begin{aligned}|(1+t\|x\|)x+t\|x\|y| &= |((1+t\|x\|)\|x\|, t\|x\|\,\|y\|)| \\ &\geqq (1+t\|x\|)\|x\|+n(|\cdot|)t\|x\|\,\|y\|\end{aligned}$$

so we conclude that $1+t\|x\|+tn(|\cdot|)\|y\|\leqq|(1,t)|$, that is,

$$\|x\|+n(|\cdot|)\|y\|\leqq\frac{|(1,t)|-1}{t}$$

and the result follows from (1.17) by letting $t\to 0$.

THEOREM 2.6. *Let M be a semi-$|\cdot|$-ideal of X. Then there is an equivalent norm on X with which M becomes a semi-M-ideal of X.*

PROOF. By Theorem 2.2 we can assume that the cotype of $|\cdot|$ is ∞, so that the type of $|\cdot|^*$ is 1 (Lemma 2.1). Since M^0 is a semi-$|\cdot|^*$-summand of X' we have by Theorem 1.11 that M^0 is a semi-L-summand of the Banach space (X', ρ_{M^0}) and it is enough to prove that ρ_{M^0} is the dual norm of a norm on X. To this end we must show that the closed unit ball of (X', ρ_{M^0}) is closed in the w^*-topology.

Let $\{h_\alpha\}$ be a net in X' satisfying $\rho_{M^0}(h_\alpha)\leqq 1$ for all α and suppose that $\{h_\alpha\}$ converges to h in the w^*-topology. We must show that $\rho_{M^0}(h)\leqq 1$. Let π be the semi-$|\cdot|^*$-projection onto M^0 and write $f_\alpha=\pi(h_\alpha)$, $g_\alpha=h_\alpha-f_\alpha$. By Theorem 1.7 we have

(a) $$\|f_\alpha\|+n(|\cdot|^*)\|g_\alpha\|=\rho_{M^0}(h_\alpha)\leqq 1$$

so $\|f_\alpha\|\leqq 1$ and $\|g_\alpha\|\leqq 1/n(|\cdot|^*)$ for all α. For a convenient subnet we can now suppose $\{f_\alpha\}\to f$, $\{g_\alpha\}\to f_0+g$ in the w^*-topology with $f, f_0\in M^0$, $g\in\operatorname{Ker}\pi$ and that $\{\|f_\alpha\|\}\to a$, $\{\|g_\alpha\|\}\to b$. We have clearly.

(b) $$h=f+f_0+g$$

and

(c) $$\|f\|\leqq a,$$

and from (a) we deduce

(d) $$a+n(|\cdot|^*)b\leqq 1.$$

For arbitrary $\varepsilon>0$ and large enough α we have $\|g_\alpha\|\leqq b+\varepsilon$ and we can apply the above Lemma to the net $\{g_\alpha/(b+\varepsilon)\}$ to obtain

$$\|f_0\|+n(|\cdot|^*)\|g\|\leqq n(|\cdot|^*)(b+\varepsilon)$$

so letting $\varepsilon \to 0$ we have

$$\text{(e)} \qquad \|f_0\| + n(|\cdot|^*)\|g\| \leqq n(|\cdot|^*)b.$$

Now from (b), (c), (d), (e) and Theorem 1.7 we deduce

$$\rho_{M^0}(h) = \|f + f_0\| + n(|\cdot|^*)\|g\| \leqq \|f\| + \|f_0\| + n(|\cdot|^*)\|g\|$$
$$\leqq a + n(|\cdot|^*)b \leqq 1,$$

as required.

REMARK 2.7. We point out for later use that the renorming processes carried out in Theorems 2.3 and 2.6 are each the converse of the other. To verify this it is enough to look at the corresponding dual norms (see Remark 2.4) and to take into account Remark 1.12. Note the close relation between the renorming processes carried out in this and the first section. The point is that when Theorem 1.11 is applied to a dual Banach space X with a w^*-closed semisummand M, then the resulting new norm $|||\cdot|||$ is dual.

Lima [16; Corollary 6.6] has proved that semi-M-ideals are proximinal subspaces and [17; Theorem 1.2] has studied the best approximation mapping onto a semi-M-ideal. Next we extend this result to arbitrary semiideals. The following Lemma is an improvement of Lemma 2.1

LEMMA 2.8. *Let $|\cdot|$ be an absolute norm. Then*

$$n(|\cdot|^*) = \operatorname{Max}\{\beta \geqq 0 : |(\beta, 1)| = 1\}.$$

PROOF. In the Banach space $(\mathbf{R}^2, |\cdot|^*)$ we have as in the proof of Lemma 2.1

$$D((1,0)) = \{(\beta, 1) : |(\beta, 1)| = 1\}.$$

Write $\alpha = \operatorname{Max}\{\beta \geqq 0 : |(\beta, 1)| = 1\}$. By Lemma 1.5 we have that $n(|\cdot|^*)$ belongs to $V((1,0),(0,1))$ so $(n(|\cdot|^*), 1)$ belongs to $D((1,0))$ and we have $n(|\cdot|^*) \leqq \alpha$. On the other hand, it is clear that $(\alpha, 1)$ belongs to $D((1,0))$ so we have again by Lemma 1.5 that $\alpha \leqq v((1,0),(0,1)) = n(|\cdot|^*)$.

COROLLARY 2.9. *Let M be a semi-$|\cdot|$-ideal of X. Then M is a proximinal subspace of X and*

$$B_M^{\text{int}}(0, 2n(|\cdot|^*)\|x + M\|) \subset P_M(x) - P_M(x) \subset B_M(0, 2n(|\cdot|^*)\|x + M\|)$$

holds for all x in X.

PROOF. In view of Theorem 2.2 and Proposition 1.2 we can suppose that the cotype of $|\cdot|$ is ∞. By Theorem 2.6 there is an equivalent norm $|||\cdot|||$ on X such that M is a semi-M-ideal of $(X, |||\cdot|||)$. By Remark 2.7 we have

$$\|x\| = |(n(|\cdot|^*)|||x|||, |||x+M|||)| \qquad (x \in X).$$

If we put in the above equality $x+m$ instead of x with x arbitrary and m in M and take greatest lower bounds of both members with m running along M, we obtain

$$\|x+M\| = |||x+M|||\,|(n(|\cdot|^*),1)| = |||x+M|||$$

where we have used the above Lemma.

For x in X and m in M we have

$$\|x-m\| = |(n(|\cdot|^*)|||x-m|||, |||x+M|||)|$$

so we can use again the above Lemma to obtain that $\|x-m\| = \|x+M\|$ if and only if $|||x-m||| = |||x+M|||$. Therefore the sets of best approximation of x in M for the norms $\|\cdot\|$ and $|||\cdot|||$ agree. Then the result follows from [17; Theorem 1.2] by taking into account that $\|m\| = n(|\cdot|^*)|||m|||$ for all m in M.

Recall that semisummands and semiideals are different generalizations of summands. The relation between both generalizations is clarified by the following result which is another application of the renorming Theorems in this section.

THEOREM 2.10. *Let M be a semi-$|\cdot|$-summand and a semi-$|\cdot|$-ideal of X. Then M is a $|\cdot|$-summand of X.*

PROOF. By Theorems 1.4 and 2.2 we can suppose that the type of $|\cdot|$ is 1 and that its cotype is ∞. To avoid ambiguities let $\|\cdot\|_1$ denote the norm of X and let $\|\cdot\|_2$ be the equivalent norm on X given by Theorem 2.6 such that M is a semi-M-ideal of $(X, \|\cdot\|_2)$. For $i = 1,2$ let V_i, v_i and $\rho_M^{(i)}$ be defined by (1.10), (1.14) and (1.18) for the corresponding norm $\|\cdot\|_i$. Recall that

$$\|x\|_1 = |(n(|\cdot|^*)\|x\|_2, \|x+M\|_2)|$$

in view of Remark 2.7. Let $m \in M$ with $\|m\|_1 = 1$ and $x \in X$ be fixed and note that $\|n(|\cdot|^*)m\|_2 = 1$. For all $\alpha > 0$ we have

(a) $$\|m+\alpha x\|_1 = |(n(|\cdot|^*)\|m+\alpha x\|_2, \alpha\|x+M\|_2)|.$$

Consider the mapping $F:[0,1] \to (\mathbf{R}^2, |\cdot|)$ defined by

$$F(\alpha) = (n(|\cdot|^*)\|m+\alpha x\|_2, \alpha\|x+M\|_2).$$

We have clearly $|F(0)| = |(1,0)| = 1$ and by (1.16) F is differentiable at zero with

(b) $$F'(0) = (\operatorname{Max}\operatorname{Re} V_2(n(|\cdot|^*)m, n(|\cdot|^*)x), \|x+M\|_2)$$

so we can apply Lemma 1.6 to F and in view of (a) and (1.16) we obtain

$$\begin{aligned}\operatorname{Max}\operatorname{Re} V_1(m,x) &= \operatorname{Max}\operatorname{Re} V(F(0), F'(0))\\ &= \operatorname{Max}\operatorname{Re}[V_2(n(|\cdot|^*)m, n(|\cdot|^*)x) + E(0, n(|\cdot|)\|x+M\|_2)]\end{aligned}$$

where for the last equality we have used Lemma 1.5. As in the proof of Theorem 1.7 we deduce

$$V_1(m,x) = V_2(n(|\cdot|^*)m, n(|\cdot|^*)x) + E(0, n(|\cdot|)\|x+M\|_2)$$

hence

$$v_1(m,x) = v_2(n(|\cdot|^*)m, n(|\cdot|^*)x) + n(|\cdot|)\|x+M\|_2.$$

Now let m run along the unit sphere of $(M, \|\cdot\|_1)$ so that $n(|\cdot|^*)m$ runs along the unit sphere of $(M, \|\cdot\|_2)$ and we obtain

(c) $$\rho_M^{(1)}(x) = n(|\cdot|^*)\rho_M^{(2)}(x) + n(|\cdot|)\|x+M\|_2.$$

Let π be the semi-$|\cdot|$-projection from $(X, \|\cdot\|_1)$ onto M. By Theorem 1.7 and Proposition 1.2 we have

(d) $$\rho_M^{(1)}(x) = \|\pi(x)\|_1 + n(|\cdot|)\|x+M\|_1.$$

As in the proof of Corollary 2.9 we have $\|x+M\|_1 = \|x+M\|_2$ so (c) and (d) give us

$$\|\pi(x)\|_1 = n(|\cdot|^*)\rho_M^{(2)}(x)$$

and we have proved that π is linear.

3. Semiidealoids

In this third section we deal with those closed subspaces M of our Banach space X such that M^0 is a semi-$|\cdot|^*$-ideal of X'. Such a subspace will be called a *semi-$|\cdot|$-idealoid* of X. We say that M is a *semiidealoid* of X when there is a (unique) absolute norm $|\cdot|$ such that M is a semi-$|\cdot|$-idealoid of X. Although for many absolute norms $|\cdot|$ (including the classical L^p norms) semi-$|\cdot|$-idealoids are either semi-$|\cdot|$-summands or semi-$|\cdot|$-ideals (see the next Theorem), in our general context the concept of semiidealoid provides a new interesting class of subspaces of a Banach space. Actually we will prove the existence of semiidealoids which are neither semisummands nor semiideals. Thus the concept of semiidealoid is probably the most suggestive one in this paper. The following Theorem summarizes the relation between the new concept and the above discussed ones.

THEOREM 3.1. *Let $|\cdot|$ be an absolute norm. Then:*

(i) *Every semi-$|\cdot|$-summand of X is a semi-$|\cdot|$-idealoid of X.*

(ii) *A closed subspace of X is a semi-$|\cdot|$-ideal and a semi-$|\cdot|$-idealoid of X if and only if it is a $|\cdot|$-ideal of X.*

(iii) *If the type of $|\cdot|$ is not 1, then every semi-$|\cdot|$-idealoid of X is actually a $|\cdot|$-ideal of X.*

(iv) *If the cotype of $|\cdot|$ is not ∞, then every semi-$|\cdot|$-idealoid of X is actually a semi-$|\cdot|$-summand of X.*

PROOF. (i) Let M be a semi-$|\cdot|$-summand of X and note that in view of Theorem 1.4 we can suppose that the type of $|\cdot|$ is 1. By Theorem 1.11 ρ_M is an equivalent norm on X such that M is a semi-L-summand of (X, ρ_M). Then by [16; Theorem 6.14] M^{00} is a semi-L-summand of (X'', ρ''_M) where ρ''_M denotes the bidual norm of ρ_M. Now we can apply Theorem 1.11 to obtain that if we write

$$|||F||| = \left|\left(\rho''_M(F), \frac{1}{n(|\cdot|)}\rho''_M(F+M^{00})\right)\right|^+ \qquad (F \in X'')$$

then $|||\cdot|||$ is an equivalent norm on X'' such that M^{00} is a semi-$|\cdot|$-summand of $(X'', |||\cdot|||)$. The proof of (i) will be concluded by showing that $|||\cdot||| = \|\cdot\|$. By Remark 1.12 (or directly from Theorem 1.7 (iii)) we have that

$$\|x\| = \left|\left(\rho_M(x), \frac{1}{n(|\cdot|)}\rho_M(x+M)\right)\right|^+ \quad (x \in X).$$

So the mapping

$$x \to \left(x, \frac{1}{n(|\cdot|)}(x+M)\right)$$

is a linear isometry from $(X, \|\cdot\|)$ into $X \times (X/M)$, the norm in the last space being given by

$$|(\rho_M(x), \rho_M(y+M))|^+ \qquad (x, y \in X).$$

Now the bitranspose mapping

$$F \to \left(F, \frac{1}{n(|\cdot|)}(F+M^{00})\right)$$

is a linear isometry from $(X'', \|\cdot\|)$ into $X'' \times (X''/M^{00})$, the norm in the last space being given by

$$|(\rho''_M(F), \rho''_M(G + M^{00}))|^+ \qquad (F, G \in X'').$$

So we have shown that

$$\|F\| = \left|\left(\rho''_M(F), \frac{1}{n(|\cdot|)}\rho''_M(F + M^{00})\right)\right|^+ = |||F|||$$

for all F in X'', as required.

(ii) Predualize Theorem 2.10.

(iii) The cotype of $|\cdot|^*$ is not ∞ (Lemma 2.1), so if M is a semi-$|\cdot|$-idealoid of X, then M^0 is a $|\cdot|^*$-summand of X' (Theorem 2.2) and M is a $|\cdot|$-ideal of X.

(iv) The linear part of this assertion is Theorem 9 in [21], the proof of which remains true without changes in our eventually nonlinear case.

REMARKS 3.2. (i) For the classical L^p norms our Theorem assures that semi-L-idealoids are semi-L-summands (this is known in [16; Theorem 6.14]) and that semi-L^p-idealoids for $p > 1$ are L^p-ideals (in fact L^p-summands unless $p = \infty$ in view of Theorem 2.2).

(ii) The restrictions on the absolute norm in statements (iii) and (iv) of the above Theorem are essential. For if the type of $|\cdot|$ is 1, then by Theorem 1.4 there are semi-$|\cdot|$-summands (so semi-$|\cdot|$-idealoids) which are not $|\cdot|$-summands (so not $|\cdot|$-ideals in view of Theorem 2.10), and on the other hand, if the cotype of $|\cdot|$ is ∞, then by Theorem 2.3 there are $|\cdot|$-ideals (so semi-$|\cdot|$-idealoids) which are not $|\cdot|$-summands (so not semi-$|\cdot|$-summands in view of Theorem 2.10).

After Theorem 3.1 the only remaining question about the relation between semisummands, semiideals and semiidealoids is the existence, for a type 1 and cotype ∞ absolute norm $|\cdot|$, of semi-$|\cdot|$-idealoids other than semi-$|\cdot|$-summands and $|\cdot|$-ideals. The rest of this section is devoted to answering this question.

For each real number γ with $0 \leqq \gamma \leqq 1$ we consider the "hexagonal" absolute norm $|\cdot|_\gamma$ defined by

$$|(a, b)|_\gamma = \operatorname{Max}\{|b|, |a| + \gamma|b|\} \qquad (a, b \in \mathbf{R}).$$

Note that $|\cdot|_0 = M$ and $|\cdot|_1 = L$, while for $0 < \gamma < 1$ the type 1 and cotype ∞ absolute norms $|\cdot|_\gamma$ will play in our situation a similar role to the ones played by L and M in sections 1 and 2 respectively.

THEOREM 3.3. *Let $0 \leqq \gamma \leqq 1$ be a fixed real number and let M, N be semi-$|\cdot|_\gamma$-idealoids of the Banach spaces X, Y respectively. Define in $X \times Y$ a* *norm by*

$$\|(x,y)\| = \mathrm{Max}\{\|x+M\|+\|y+N\|, \|x\|+\gamma\|y+N\|, \|y\|+\gamma\|x+M\|\}.$$

Then $M\times N$ is a semi-$|\cdot|_\gamma$-idealoid of $X\times Y$. Also $M\times N$ is a semi-$|\cdot|_\gamma$-summand (resp. $|\cdot|_\gamma$-ideal) of $X\times Y$ if and only if M and N are semi-$|\cdot|_\gamma$-summands (resp. $|\cdot|_\gamma$-ideals) of X and Y respectively.

PROOF. (i) First we prove that if M and N are semi-$|\cdot|_\gamma$-summands of X and Y respectively, then $M\times N$ is a semi-$|\cdot|_\gamma$-summand of $X\times Y$. More concretely, let π_M (resp. π_N) be the semi-$|\cdot|_\gamma$-projection from X (resp. Y) onto M (resp. N) and define

$$\pi(x,y) = (\pi_M(x), \pi_N(y)) \qquad (x\in X, y\in Y).$$

Then π is clearly a semiprojection on $X\times Y$ whose range is $M\times N$. A straightforward computation shows that π is a semi-$|\cdot|_\gamma$-projection (the norm on $X\times Y$ has been happily found to this end).

(ii) Now let M and N be arbitrary semi-$|\cdot|_\gamma$-idealoids of X and Y respectively. Then M^{00} and N^{00} are semi-$|\cdot|_\gamma$-summands of X'' and Y''. By (i) we have that $M^{00}\times N^{00}$ is a semi-$|\cdot|_\gamma$-summand of $X''\times Y''$, the norm on $X''\times Y''$ being defined by

$$\|(F,G)\| = \mathrm{Max}\{\|F+M^{00}\|+\|G+N^{00}\|, \|F\|+\gamma\|G+N^{00}\|, \|G\|+\gamma\|F+M^{00}\|\}.$$

But the bidual space of $X\times Y$ is identified in a natural way with $X''\times Y''$ with the above norm and in this identification the bipolar of $M\times N$ is just $M^{00}\times N^{00}$, so $M\times N$ is a semi-$|\cdot|_\gamma$-idealoid of $X\times Y$.

(iii) Assume that $M\times N$ is a semi-$|\cdot|_\gamma$-summand of $X\times Y$. In order to prove that M and N are semi-$|\cdot|_\gamma$-summands of X and Y, let π be the semi-$|\cdot|_\gamma$-projection from $X\times Y$ onto $M\times N$, fix x in X and write $\pi(x,0)=(m_0,n_0)$. It is enough to show that $n_0=0$. Since $\|(x,0)+M\times N\| = \|x+M\|$ we have that

$$\begin{aligned} P_{M\times N}((x,0)) &= \{(m,n)\in M\times N : \|(x-m,-n)\| = \|x+M\|\} \\ &= \{(m,n)\in M\times N : \|x-m\| = \|x+M\|, \|n\| \leqq (1-\gamma)\|x+M\|\} \\ &= P_M(x)\times B_N(0,(1-\gamma)\|x+M\|). \end{aligned}$$

On the other hand we have by Proposition 1.2 that

$$\begin{aligned} P_{M\times N}((x,0)) &= B_{M\times N}((m_0,n_0),(1-\gamma)\|x+M\|) \\ &= B_M(m_0,(1-\gamma)\|x+M\|)\times B_N(n_0,(1-\gamma)\|x+M\|). \end{aligned}$$

Compare the above two equalities to obtain $n_0=0$, as required.

(iv) If $M \times N$ is a $|\cdot|_\gamma$-ideal of $X \times Y$, then $M^{00} \times N^{00}$ is a $|\cdot|_\gamma$-summand of $X'' \times Y''$ and the corresponding $|\cdot|_\gamma$-projection is w^*-continuous. It follows from (iii) that M^{00} is a $|\cdot|_\gamma$-summand of X'' and that the $|\cdot|_\gamma$-projection from X'' onto M^{00} is w^*-continuous. Therefore M is a $|\cdot|_\gamma$-ideal of X and the same argument applies to N.

REMARK 3.4. The above Theorem together with previous results give the existence for each $0 < \gamma < 1$ of a Banach space with a semi-$|\cdot|_\gamma$-idealoid which is neither a semi-$|\cdot|_\gamma$-summand nor a $|\cdot|_\gamma$-ideal. By Theorem 1.4 there is a Banach space X with a semi-$|\cdot|_\gamma$-summand M which is not a $|\cdot|_\gamma$-summand and by Theorem 2.3 there is a Banach space Y with a $|\cdot|_\gamma$-ideal N which is not a $|\cdot|_\gamma$-summand. By Theorems 3.1 and 3.3 $M \times N$ is a semi-$|\cdot|_\gamma$-idealoid of $X \times Y$ with suitable norm. If $M \times N$ were a semi-$|\cdot|_\gamma$-summand of $X \times Y$, then by Theorems 3.3 and 2.10 N would be a $|\cdot|_\gamma$-summand of Y, a contradiction. If $M \times N$ were a $|\cdot|_\gamma$-ideal of $X \times Y$, then the same arguments show that M would be a $|\cdot|_\gamma$-summand of X, a contradiction.

In what follows we use a renorming process to obtain for each type 1 and cotype ∞ absolute norm $|\cdot|$ semi-$|\cdot|$-idealoids which are neither semi-$|\cdot|$-summands nor $|\cdot|$-ideals.

For the next Lemma the definition of a seminorm ρ_M associated to any nonzero subspace M of a Banach space X should be recalled (see section 1). In the proof a space will be considered as a subspace of two different Banach spaces and we emphasize if necessary this fact by writing $\rho_{(X,M)}$ instead of ρ_M. It is clear that if $M \subset Y \subset X$, then $\rho_{(Y,M)}(y) = \rho_{(X,M)}(y)$ for all y in Y.

LEMMA 3.5. *Let M be a semi-$|\cdot|$-idealoid of X. Then*

$$\rho_M(x) = \rho_{M^{00}}(J_X(x))$$

and

$$\|x\| = |(\rho_M(x), \|x + M\|)|^+ \qquad \textit{for all } x \textit{ in } X.$$

PROOF. Let π be the semi-$|\cdot|$-projection from X'' onto M^{00}. For x in X and m in M with $\|m\| = 1$ we have by Theorem 1.7(i) that

$$V(m,x) = V(J_X(m), J_X(x)) = V(J_X(m), \pi J_X(x)) + E(0, n(|\cdot|)\|x + M\|)$$

so

$$\rho_M(x) = \rho_{(X,M)}(x) = \rho_{(X'',J_X(M))}(\pi J_X(x)) + n(|\cdot|)\|x + M\|$$
$$= \rho_{(M^{00},J_X(M))}(\pi J_X(x)) + n(|\cdot|)\|x + M\|.$$

Since the natural identification of M^{00} with M'' maps $J_X(M)$ onto $J_M(M)$ it follows from Lemma 1.16 that

$$\rho_{(M^{00},J_X(M))}(\pi J_X(x)) = \|\pi J_X(x)\|.$$

Thus we have

$$\rho_M(x) = \|\pi J_X(x)\| + n(|\cdot|)\|x + M\|$$

and the equality $\rho_M(x) = \rho_{M^{00}}(J_X(x))$ follows from Theorem 1.7(iii).

The equality $\|x\| = |(\rho_M(x), \|x + M\|)|^+$ follows from the first part of the Lemma and the fact that this equality is true in case M is a semi-$|\cdot|$-summand of X (apply Theorem 1.7(iii)).

COROLLARY 3.6. *Let $|\cdot|$ be a type 1 absolute norm and M a semi-$|\cdot|$-idealoid of X. Then ρ_M is an equivalent norm on X. Let ρ''_M denote the bidual norm of ρ_M on X''. Then $\rho_{M^{00}}(F) \leqq \rho''_M(F)$ for all F in X''. Moreover, if $\rho_{M^{00}}(F) < \rho''_M(F)$ holds for some F in X'', then F satisfies the inequality $\rho''_M(F) \leqq (n(|\cdot|) + n(|\cdot|^*))\|F + M^{00}\|$.*

PROOF. Since M^{00} is a semi-$|\cdot|$-summand of X'' we have by Theorem 1.11 that $\rho_{M^{00}}$ is an equivalent norm on X'' and the first equality in the above Lemma gives us that ρ_M is an equivalent norm on X.

Fix F in X'' and let $\{x_\alpha\}$ be a net in X satisfying $\rho_M(x_\alpha) \leqq \rho''_M(F)$ and such that the net $\{J_X(x_\alpha)\}$ converges to F in the w^*-topology of X''. By the w^*-lower semicontinuity of the norm $\rho_{M^{00}}$ (see Remark 2.7) and the first equality in the above Lemma we have that

$$\rho_{M^{00}}(F) \leqq \lim \operatorname{Inf}\{\rho_{M^{00}}(J_X(x_\alpha))\} = \lim \operatorname{Inf}\{\rho_M(x_\alpha)\} \leqq \rho''_M(F).$$

By a similar argument to the one used in the proof of the first part of Theorem 3.1 we can bidualize the second equality in Lemma 3.5 to obtain

$$\|F\| = |(\rho''_M(F), \|F + M^{00}\|)|^+ \qquad (F \in X'').$$

On the other hand we can directly apply to M^{00} and X'' the second part of Lemma 3.5 and we obtain

$$\|F\| = |(\rho_{M^{00}}(F), \|F + M^{00}\|)|^+ \qquad (F \in X'').$$

Then if $F \in X''$ is such that $\rho_{M^{00}}(F) < \rho''_M(F)$ we can apply Lemmas 1.1 and 2.8 and we obtain

$$\rho''_M(F) \leqq n(|\cdot|^{+*})\|F + M^{00}\|.$$

The desired inequality follows from the fact that $n(|\cdot|^{+*}) = n(|\cdot|) + n(|\cdot|^*)$ which can be easily verified by using Lemma 1.10.

THEOREM 3.7. *Let* ${}_0|\cdot|$ *and* $|\cdot|$ *be type* 1 *absolute norms satisfying* $n_0^*/n_0 \leqq n^*/n$ *where* $n_0^* = n({}_0|\cdot|^*)$, $n_0 = n({}_0|\cdot|)$, $n = n(|\cdot|)$ *and* $n^* = n(|\cdot|^*)$. *Let M be a semi-*${}_0|\cdot|$*-idealoid of X and define*

$$|||x||| = \left|\left(\rho_M(x), \frac{n_0}{n}\|x+M\|\right)\right|^+ \qquad (x \in X).$$

Then $|||\cdot|||$ *is an equivalent norm on X and M is a semi-*$|\cdot|$*idealoid of* $(X, |||\cdot|||)$. *Moreover, if M is a semi-*${}_0|\cdot|$*-summand (resp.* ${}_0|\cdot|$*-ideal) of* $(X, \|\cdot\|)$, *then M is a semi-*$|\cdot|$*-summand (resp.* $|\cdot|$*-ideal) of* $(X, |||\cdot|||)$.

PROOF. Since M^{00} is a semi-${}_0|\cdot|$-summand of X'' it follows from Theorem 1.11 that if we write

$$q(F) = \left|\left(\rho_{M^{00}}(F), \frac{n_0}{n}\|F+M^{00}\|\right)\right|^+ \qquad (F \in X'')$$

then q is an equivalent norm on X'' such that M^{00} is a semi-$|\cdot|$-summand of (X'', q). So for the first part of the Theorem it is enough to show that $|||F||| = q(F)$ for all F in X''.

By bidualizing the defining equality of $|||\cdot|||$ we have that

$$|||F||| = \left|\left(\rho''_M(F), \frac{n_0}{n}\|F+M^{00}\|\right)\right|^+ \qquad (F \in X'')$$

and the equality $|||F||| = q(F)$ is clear when $\rho_{M^{00}}(F) = \rho''_M(F)$. Otherwise by the above Corollary and our assumption that $n_0^*/n_0 \leqq n^*/n$ we have that

$$\rho_{M^{00}}(F) < \rho''_M(F) \leqq (n_0 + n_0^*)\|F+M^{00}\| \leqq (n+n^*)\frac{n_0}{n}\|F+M^{00}\|$$

$$= n(|\cdot|^{+*})\frac{n_0}{n}\|F+M^{00}\|$$

and it is enough to apply Lemma 2.8 to obtain that $|||F||| = q(F)$.

If M is actually a semi-${}_0|\cdot|$-summand of $(X, \|\cdot\|)$, then M is a semi-$|\cdot|$-summand of $(X, |||\cdot|||)$ (Theorem 1.11), while if M is a ${}_0|\cdot|$-ideal of $(X, \|\cdot\|)$ then we can apply Theorem 1.11 to obtain that M^{00} is a $|\cdot|$-summand of $(X'', |||\cdot|||)$ with w^*-continuous $|\cdot|$-projection and therefore M is a $|\cdot|$-ideal of $(X, |||\cdot|||)$.

REMARKS 3.8. (i) Assume in the above Theorem that $n_0^*/n_0 = n^*/n$. Then we can apply again the Theorem to the Banach space $(X, |||\cdot|||)$ resulting from it by interchanging the absolute norms ${}_0|\cdot|$ and $|\cdot|$. In this way we reencounter the initial Banach space $(X, \|\cdot\|)$. To see this it is enough to apply Remark 1.12 to X''

and M^{00}, once we know by the above proof that the norm $|||\cdot|||$ on X'' which appears when we apply to it Theorem 1.11 is just the bidual norm of the norm $|||\cdot|||$ on X defined in the above Theorem.

(ii) For shortness a semi-$|\cdot|$-idealoid will be called "proper" if it is neither a semi-$|\cdot|$-summand nor a $|\cdot|$-ideal. Note that a proper semi-$|\cdot|$-idealoid can not be a semi-$|\cdot|$-ideal by the second part of Theorem 3.1. The same Theorem shows that if either the type of $|\cdot|$ is not 1 or its cotype is not ∞, then there are no proper semi-$|\cdot|$-idealoids. Now we can exhibit a proper semi-$|\cdot|$-idealoid for each type 1 and cotype ∞ absolute norm $|\cdot|$. In fact, let $|\cdot|$ be such an absolute norm and let $_0|\cdot|$ be the hexagonal norm $|\cdot|_\gamma$ where $\gamma = n/(n+n^*)$. Since $0<\gamma<1$ there is a Banach space with a proper semi-$|\cdot|_\gamma$-idealoid (Remark 3.4). Since $n_0=\gamma$ and $n_0^*=1-\gamma$, we have that $n_0^*/n_0 = n^*/n$ so Theorem 3.7 and the above Remark are applicable and they yield the desired Banach space with a proper semi-$|\cdot|$-idealoid.

4. The concluding Theorem

For the purpose of this last section a brief summary of the relation between the three kinds of subspaces discussed in this paper should be in place. The first generation of subspaces is the one of semisummands whose linear parts are the summands. Unless the absolute norm $|\cdot|$ is of type 1 every semi-$|\cdot|$-summand is a $|\cdot|$-summand (Corollary 1.8). Conversely for each type 1 absolute norm $|\cdot|$ there are semi-$|\cdot|$-summands which are not $|\cdot|$-summands (Theorem 1.4). The second generation is the one of semiideals (closed subspaces whose polars are semisummands) whose linear parts are the ideals (closed subspaces whose polars are summands). The intersection of the first and second generation is the linear part of the first one (Theorem 2.10). Unless the cotype of the absolute norm $|\cdot|$ is ∞ every semi-$|\cdot|$-ideal is a $|\cdot|$-summand (Theorem 2.2). Conversely, for each cotype ∞ absolute norm $|\cdot|$ there are semi-$|\cdot|$-ideals which are not $|\cdot|$-ideals and $|\cdot|$-ideals which are not $|\cdot|$-summands (Theorem 2.3). The third generation is the one of semiidealoids (closed subspaces whose polars are semiideals). The third generation includes the first one (Theorem 3.1(i)) and intersects the second one in its linear part (Theorem 3.1(ii)). Unless the type of $|\cdot|$ is 1 (resp. the cotype is ∞) every semi-$|\cdot|$-idealoid is a $|\cdot|$-ideal (resp. a semi-$|\cdot|$-summand), so proper semi-$|\cdot|$-idealoids can only occur when the type of $|\cdot|$ is 1 and its cotype is ∞ (Theorem 3.1(iii) and (iv)). Conversely, we have proved that such a proper semi-$|\cdot|$-idealoid always exist when the type of $|\cdot|$ is 1 and its cotype is ∞ (Remark 3.8(ii)).

In order to complete the picture we conclude this paper by showing that the fourth generation (those closed subspaces whose polars are semiidealoids) agrees with the second one (the semiideals). More suggestively, every w^*-closed semiidealoid of a dual Banach space is a semisummand. This is agreeable news because it shows that the procedure of obtaining new concepts by consecutive predualization of the one of semisummand is achieved with the third generation. As a consequence we obtain that every w^*-closed ideal of a dual Banach space is a summand so that the first two generations are enough in the linear case. This is the reason why the concept of $|\cdot|$-idealoid was not introduced.

THEOREM 4.1. *Let M be a closed subspace of a Banach space X and let $|\cdot|$ be an absolute norm. Then M is a semi-$|\cdot|$-ideal of X if and only if M^0 is a semi-$|\cdot|^*$-idealoid of X'.*

PROOF. The "only if" part follows from Theorem 3.1(i). If the cotype of $|\cdot|$ is not ∞ the "if" part follows from Theorem 3.1(iii) and [21; Theorem 9], so we assume that the cotype of $|\cdot|$ is ∞.

Let M^0 be a semi-$|\cdot|^*$-idealoid of X'. Then M^{00} is a semi-$|\cdot|$-ideal of X'' and by Theorem 2.6 there is an equivalent norm $|||\cdot|||$ on X'' such that M^{00} is a semi-M-ideal of $(X'', |||\cdot|||)$. Moreover (Remark 2.7) this new norm is related with the old one by the equality

(a) $$\|F\| = |(n^* |||F|||, |||F + M^{00}|||)| \qquad (F \in X'')$$

where $n^* = n(|\cdot|^*)$. As in the proof of Corollary 2.9 we have $|||F + M^{00}||| = \|F + M^{00}\|$ for all F in X'', so

(b) $$\|F\| = |(n^* |||F|||, \|F + M^{00}\|)| \qquad (F \in X'').$$

Write $q(x) = |||J_X(x)|||$. Then q is an equivalent norm on X satisfying

(c) $$\|x\| = |(n^* q(x), \|x + M\|)| \qquad (x \in X)$$

(apply (b) with $F = J_X(x)$). Bidualize equality (c) to obtain

(d) $$\|F\| = |(n^* q(F), \|F + M^{00}\|)| \qquad (F \in X'')$$

(the bidual norm of q on X'' is again denoted by q). From (a) and Lemma 2.8 we have that

(e) $$\|F\| \leqq |(n^*, 1)|\; |||F||| = |||F||| \qquad \text{for all } F \text{ in } X''.$$

In particular $\|x\| \leqq q(x)$ for all x in X, so

(f) $$\|F\| \leqq q(F) \qquad \text{for all } F \text{ in } X''.$$

By applying Lemmas 1.1 and 2.8 we obtain from inequalities (b), (d), (e) and (f) that

$$|||F||| = q(F) \qquad \text{for all } F \text{ in } X''.$$

In this way we have proved that M^{00} is a semi-M-ideal of (X'', q). In view of Remark 3.2(i) we deduce that M is a semi-M-ideal of (X, q).

From (c) we deduce that

$$\|x + M\| = |(n^* q(x + M), \|x + M\|)| \qquad \text{for all } x \text{ in } X,$$

and Lemma 2.8 gives $q(x + M) \leqq \|x + M\|$, the converse inequality being clear from $q(x) \geqq \|x\|$. Now (c) reads

$$\|x\| = |(n^* q(x), q(x + M))| \qquad (x \in X)$$

and it is enough to apply Theorem 2.3 to the Banach space (X, q) which contains M as a semi-M-ideal to conclude that M is a semi-$|\cdot|$-ideal of $(X, \|\cdot\|)$.

COROLLARY 4.2. *Let M be a closed subspace of a Banach space X and let $|\cdot|$ be an absolute norm. Then M is a $|\cdot|$-ideal of X if and only if M^0 is a $|\cdot|^*$-ideal of X'.*

CONCLUDING REMARK. If follows from the results in this paper that the concept of semiidealoid becomes a new interesting topic in Geometry of Banach Spaces. The authors have obtained some relevant results on semiidealoids. For example they are proximinal subspaces. This can be proved by using Theorem 3.7 which reduces the problem (as in Remark 3.8(ii)) to the case of semi-$|\cdot|_\gamma$-idealoids (where $|\cdot|_\gamma$ is an hexagonal absolute norm) which are just those semiidealoids satisfying the $1\frac{1}{2}$-ball property (see [24]). Actually a stronger result can be obtained. Concretely the assertion of Corollary 2.9 is true for semiidealoids. We intend to deal with these and further topics in a subsequent paper.

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